1. *Let $a, b \in \mathbb{R}$ with $a < b$, let $f : (a, b) \to \mathbb{R}$ be a continuous function. Discuss the vaidity of the statement: f is uniformly continuous if and only if it is bounded. Justify your answer.*

   Solution: Since given f is uniformly continuous it will map a cauchy sequence to another cauchy sequence. Since (a, b) is dense in $[a, b]$ there exists sequences $\{x_n\}$ and $\{y_n\}$ in (a, b) such that $lim_{n\to\infty} x_n = a$ and $lim_{n\to\infty} y_n = b$. Thus, both $\{x_n\}$ and $\{y_n\}$ are cauchy sequences and so are $\{f(x_n)\}$ and $\{f(y_n)\}$.

   Hence both $\{f(x_n)\}$ and $\{f(y_n)\}$ are convergent. Let $lim_{n\to\infty} f(x_n) = c$ and $lim_{n\to\infty} f(y_n) = d$.
   Now, let us define a new function $g : [a, b] \to \mathbb{R}$ as

$$g(x) = \begin{cases} c, & \textbf{if } x = a \\ f(x), & \textbf{if } x \in (a, b) \\ d, & \textbf{if } x = b \end{cases} \tag{1}$$

   Then we get that $g : [a, b] \to \mathbb{R}$ is a continuous function and hence bounded. Thus $f(x)$ is also bounded.

   The other way is not true as $sin(\frac{1}{x})$ is a bounded continuous function on $(0, 1)$ but it is not uniformly continuous. □

2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, satisfying $lim_{x\to\infty} f = 0$ and $lim_{x\to-\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and f attains a maximum or minimum on $\mathbb{R}$. Give examples to show that f attains a maximum but not a minimum and vice versa.*

   Solution: Since $lim_{x\to\infty} f = 0$, given $\epsilon > 0$ we get $M_1 \in \mathbb{N}$ such that.

$$| f(x) | < \epsilon, \quad \forall x > M_1.$$

   again, $lim_{x\to-\infty} f = 0$ implies given $\epsilon > 0$ we get $M_2 \in \mathbb{N}$ such that.

$$| f(x) | < \epsilon, \quad \forall x < -M_2.$$

   for our case let us choose $\epsilon = 1$.

   Now since f is continuous on $\mathbb{R}$ we get that f is bounded on $[-M_2, M_1]$. Hence, we get that $| f(x) | \leq m, \forall x \in [-M_2, M_1]$ for some $m \in \mathbb{R}$ so if $M = max\{1, m\}$ then $| f(x) | \leq M$, for $x \in \mathbb{R}$ and hence is bounded.
   $f(x)$ can attain both the extremums , let

$$f(x) = \begin{cases} \frac{1}{x}, & \textbf{if } x > 1 \textbf{ or } x < -1 \\ x, & \textbf{if } x \in [-1, 1] \end{cases} \tag{2}$$

   then $f(x)$ is a continuous function such that $lim_{x\to\infty} f = 0$ and $lim_{x\to-\infty} f = 0$ and attains both the extremums 1 and -1.

Examples: $f(x) = \frac{1}{1+x^2}$ **is continuous function, satisfying** $lim_{x \to \infty} f = 0$ **and** $lim_{x \to -\infty} f = 0$**, and is bounded by** 1 **and attains only its maximum at** $x = 0$ **and do not attain its minimum which is** 0

$g(x) = -\frac{1}{1+x^2}$ **is a continuous function on** $\mathbb{R}$ **satisfying** $lim_{x \to \infty} f = 0$ **and** $lim_{x \to -\infty} f = 0$**, it attains its minimum at** $x = 0$ **but do not atain its maximum which is** 0**.** □

3. *Does there exists a continuous function which takes every real value exactly twice?Justify your answer.*

   Solution:Let us hold there is such a function. According to the assumption there are only two zeroes of f **say** x **and** $y \in \mathbb{R}$ **such that** $x < y$**, and** $f(x) = 0 = f(y)$**.**
   Let us look at the behaviour of the function on $[x, y]$**.**
   First, $f(x)$ **can either be positive or negative on** $[x, y]$**. Otherwise it will have another zero in** $[x, y]$ **by the intermediate value property, which will pose a contradiction to the assumption.**
   without loss of generality lets us assume that $f(x)$ **is positive throughout** $[x, y]$**. Since** f **is continuous it will have a maximum in** $[x, y]$ **which can't be zero as then** f **will have three zeroes and again it will give a contradiction.**
   Say there exists a $c \in (x, y)$ **such that** $f(c) = M$**.**
   Now by the assumption there is another point say $d \neq x, y$ **such that** $f(d) = M$**.**
   Now let us look at the postion of d **with respect to the interval** $[x, y]$**.**
   if $d > y$ **then the values in** $(0, M)$ **will be taking thrice , i.e. in between** $(x, c)(c, y)$ **and** (y, d)**, which will be a contradiction. Similarly,** d **can't be less than** x**.**
   d **can't be in between** (c, y) **because then in the interval** (c, d) $f(x) < M$ **which is a contradiction because the values which are less tha** M **are already been taken twice in the intervals** (a, c) **and** (d, y)**.**
   Similarly, d **can't be in between** (x, c)**. Thus such a** d **cannot exist, which tells that such a continuous function cannot exist.** □

4. *Let* $f : (0, \infty) \to \mathbb{R}$ *be a function which is differentiable on* $(0, \infty)$*.Suppose* $lim_{x \to \infty} f(x)$ *and* $lim_{x \to \infty} f'(x)$ *both exists (as real numbers).Prove that* $lim_{x \to \infty} f'(x) = 0$*.*

   Solution: Let $lim_{x \to \infty} f(x) = l$**, and** $lim_{x \to \infty} f'(x) = m$**. Thus,**

$$lim_{x \to \infty} f(x) + f'(x) = l + m.$$

   Now,

$$\begin{aligned} lim_{x \to \infty} f(x) &= lim_{x \to \infty} \frac{\exp(x) f(x)}{\exp(x)} \\ &= lim_{x \to \infty} \frac{\exp(x)(f(x) + f'(x))}{\exp(x)} \quad \textbf{(by LHospitals Rule)} \\ &= lim_{x \to \infty} (f(x) + f'(x)) \end{aligned}$$

   Hence, we get that $lim_{x \to \infty} f'(x) = 0$**.** □

5. *Let* $f : I \to \mathbb{R}$ *be a function on an interval* I *having the intermediate value property (i.e. for any two points* $a, b \in I$ *and any real number* k *between* $f(a)$ *and* $f(b)$*, there exists some* c *between* a *and* b *such that* $f(c) = k$*). Is f necessarily continuous? Justify your answer.*

Solution: Let $f(x) : [0,1] \to \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} sin(\frac{1}{x}), & \textbf{if } x \neq 0 \\ 0, & \textbf{if } x = 0 \end{cases} \tag{3}$$

Then $f(x)$ is surely not continuous at 0, but $f(x)$ has the intermediate value property. □

6. *Let f be continuous on $[a,b]$ and assume the second derivative f'' exists on (a,b). Suppose the graph of f and the line joining (a,f(a)) and (b,f(b)) intersect at a point (c,f(c)) where $a < c < b$. Show that there exists a point d such that $f''(d) = 0$.*

   Solution: Since the the points $(a, f(a))(b, f(b)), (c, f(c))$ lie on the same line, i.e. the line joining $(a, f(a)), (b, f(b))$, we get that the slope of the line joining $(a, f(a))(c, (fc))$ and $(c, f(c))(b, f(b))$ are equal, i.e.

   $$\frac{(f(c) - f(a))}{(c - a)} = \frac{(f(b) - f(c))}{(b - c)}$$

   By Lagrange's mean value theorem we get that there are two points $m \in (a, c)$ and $n \in (c, b)$ such that

   $$f'(m) = \frac{(f(c) - f(a))}{(c - a)}$$

   and

   $$f'(n) = \frac{(f(b) - f(c))}{(b - c)}$$

   Since, f'' exists on(a, b) we get that f' is continuous on (a, b). Hence, if we apply the Rolle's theorem in the interval $[m, n]$ we get that there is a point $d \in (m, n)$ such that $f''(d) = 0$. □

7. *Does the limit $lim_{x \to 0+}(sin(x)^x)$ exist ? If yes what is it?*

   Solution: Let $y = (sin(x)^x)$ for $x > 0$. Then $\ln y = x \ln sin(x)$ then we have :

   $$\mid sin(x) \ln sin(x) \mid < \mid x \ln sin(x) \mid < \mid x \ln x \mid$$

   now,

   $$\begin{aligned} lim_{x \to 0+} x \ln x &= lim_{x \to 0+} \frac{\ln x}{\frac{1}{x}} \\ &= lim_{x \to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \text{ by LHospitals Rule} \\ &= lim_{x \to 0+} - x \\ &= 0. \end{aligned}$$

   hence, we also get that $lim_{x \to 0+} sin(x) \ln sin(x) = 0$. thus, by sandwich theorem we see that $lim_{x \to 0+} x \ln sin(x) = 0$.
   So,

   $$lim_{x \to 0+} y = lim_{x \to 0+} \exp(x \ln sin(x)) = \exp(0) = 1.$$

   Hence, $lim_{x \to 0+}(sin(x)^x)$ exists and is equal to 1.